6.1 Wells' Chapter VII

VII. SPECIAL METHODS IN MULTIPLICATION AND DIVISION

128. **Any Power of a Power.**

Required the value of $(a^m)^n$, where m and n are any positive integers.

We have,

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times \ldots \text{to } n \text{ factors } (\S 60)\\ &= a^{m+m+\ldots \text{ to } n \text{ terms}} = a^{mn}.\end{aligned}$$

129. **Any Power of a Product.**

Required the value of $(abc\ldots)^n$, where n is any positive integer.

We have,

$$\begin{aligned}(abc\ldots)^n &= (abc\ldots) \times (abc\ldots) \times \ldots \text{ to } n \text{ factors}\\ &= (a \times a \times \ldots \text{ to } n \text{ factors})(b \times b \times \ldots \text{ to } n \text{ factors})\ldots\\ &= a^n b^n c^n \ldots.\end{aligned}$$

130. **Any Power of a Monomial.**

1. Required the value of $\left(5a^2b\right)^3$.

We have, $\left(5a^2b\right)^3 = 5a^2b \times 5a^2b \times 5a^2b = 125a^6b^3$.

2. Required the value of $(-m)^4$.

We have, $(-m)^4 = (-m) \times (-m) \times (-m) \times (-m) = m^4$.

3. Required the value of $\left(-3n^3\right)^3$.

We have, $\left(-3n^3\right)^3 = \left(-3n^3\right) \times \left(-3n^3\right) \times \left(-3n^3\right) = -27n^9$.

From §§128 and 129, and the above examples, we have the following rule for raising a rational and integral monomial (§63) to any power whose exponent is a positive integer:

Raise the absolute value of the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Give to every power of a positive term, and to every **even** *power of a negative term, the positive sign; and to every* **odd** *power of a negative term the negative sign.*

131. **Square of a Binomial.**

We find by actual multiplication,

$$(a+b)^2 = (a+b) \times (a+b) = a^2 + 2ab + b^2, \quad (1)$$
$$(a-b)^2 = (a-b) \times (a-b) = a^2 - 2ab + b^2. \quad (2)$$

That is,

The square of the sum of two numbers equals the square of the first, plus twice the product of the first by the second, plus the square of the second.

The square of the difference of two numbers equals the square of the first, minus twice the product of the first by the second, plus the square of the second.

In the remainder of the book, we shall, for the sake of brevity, use the expression "*the difference of a and b*" to denote the *remainder obtained by subtracting b from a.*

1. Square $3a^2 - 2b$.

   By (2), $\left(3a^2 - 2b\right)^2 = \left(3a^2\right)^2 - 2\left(3a^2\right)(2b) + (2b)^2 = 9a^4 - 12a^2b + 4b^2$ (§130).

   If the first term of the binomial is negative, it should be enclosed, negative sign and all, in parentheses, before applying the rule.

2. Square $-4x^3 + 8$.

$$\begin{aligned}\left(-4x^3+9\right)^2 &= \left[\left(-4x^3\right)+9\right]^2\\ &= \left(-4x^3\right)^2 + 2\left(-4x^3\right)(9) + 9^2,\ \text{by}(1)\\ &= 16x^6 - 72x^3 + 81\end{aligned}$$

132. **Product of the Sum and Difference of Two Numbers.**

We find by actual multiplication,

$$(a+b)(a-b) = a^2 - b^2.$$

That is, *the product of the sum and difference of two numbers equals the difference of their squares.*

1. Multiply $6a + 5b^3$ by $6a - 5b^3$.

   By the rule,

$$\left(6a+5b^3\right)\left(6a-5b^3\right) = (6a)^2 - \left(5b^3\right)^2 = 36a^2 - 25b^6.$$

2. Multiply $-x^2 + 4$ by $-x^2 - 4$.

$$\begin{aligned}\left(-x^2+4\right)\left(-x^2-4\right) &= \left[\left(-x^2\right)+4\right]\left[\left(-x^2\right)-4\right]\\ &= \left(-x^2\right)^2 - 4^2 = x^4 - 16.\end{aligned}$$

3. Expand $(a+b-c)(a-b+c)$.

   To *expand* an algebraic expression is to perform the operations indicated.

By §82,

$$\begin{aligned}(a+b-c)(a-b+c) &= [a+(b-c)][a-(b-c)] \\ &= a^2-(b-c)^2, \text{ by the rule,} \\ &= a^2-\left(b^2-2bc+c^2\right) \\ &= a^2-b^2+2bc-c^2.\end{aligned}$$

4. Expand $(x+y+z)(x-y+z)$.

$$\begin{aligned}(x+y+z)(x-y+z) &= [(x+z)+y][(x+z)-y] \\ &= (x+z)^2-y^2 \\ &= x^2+2xz+z^2-y^2.\end{aligned}$$

133. **Product of Two Binomials having the Same First Term.**

We find by actual multiplication

$$(x+a)(x+b) = x^2+(a+b)x+ab.$$

That is,

The product of two binomials having the same first term equals the square of the first term, plus the algebraic sum of the second terms multiplied by the first term, plus the product of the second terms.

1. Multiply $x-5$ by $x+3$.

   By the above rule, the coefficient of x is the sum of -5 and $+3$, or -2, and the last term is the product of -5 and $+3$ or -15

   Whence, $(x-5)(x+3) = x^2-2x-15$.

2. Multiply $x-5$ by $x-3$.

   The coefficient of x is the sum of -5 and -3, or -8, and the last term is the product of -5 and -3, or 15

   Whence, $(x-5)(x-3) = x^2-8x+15$.

3. Multiply $ab-4$ by $ab+7$.

   By the rule, $(ab-4)(ab+7) = a^2b^2+3ab-28$.

4. Multiply $m+n+6$ by $m+n+8$.

$$\begin{aligned}(m+n+6)(m+n+8) &= [(m+n)+6][(m+n)+8] \\ &= (m+n)^2+14(m+n)+48.\end{aligned}$$

134. **Square of a Polynomial.**

By §131,(1),

$$(a_1 + a_2)^2 = a_1^2 + a_2^2 + 2a_1a_2 \tag{1}$$

We also have,

$$\begin{aligned}(a_1 + a_2 + a_3)^2 &= [(a_1 + a_2) + a_3]^2 \\ &= (a_1 + a_2)^2 + 2(a_1 + a_2) \times a_3 + a_3^2 \\ &= a_1^2 + 2a_1a_2 + a_2^2 + 2a_1a_3 + 2a_2a_3 + a_3^2 \\ &= a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3\end{aligned}$$

$$(a_1 + a_2 + a_3)^2 = a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3 \tag{2}$$

The results (1) and (2) are in accordance with the following law :

The square of a polynomial equals the sum of the squares of its terms, plus twice the product of each term by each of the following terms.

We will now prove that this law holds for the square of any polynomial.

Assume that the law holds for the square of a polynomial of m terms, where m is any positive integer; that is, $(a_1 + a_2 + a_3 + \cdots + a_{m-1} + a_m)^2$

$$\begin{aligned}= \; & a_1^2 + a_2^2 + \cdots + a_m^2 + 2a_1(a_2 + \cdots + a_m) \\ & +2a_2(a_3 + \cdots + a_m) + \cdots + 2a_{m-1}a_m.\end{aligned} \tag{3}$$

Then, $(a_1 + a_2 + a_3 + \cdots + a_m + a_{m+1})^2$

$$\begin{aligned}&= [(a_1 + a_2 + \cdots + a_m) + a_{m+1}]^2 \\ &= (a_1 + a_2 + \cdots + a_m)^2 \\ &\qquad + 2(a_1 + a_2 + \cdots + a_m)a_{m+1} + {a_{m+1}}^2, \text{ by (1)} \\ &= a_1^2 + {a_2}^2 + \cdots + {a_m}^2 + {a_{m+1}}^2 \\ &\quad + 2a_1(a_2 + \cdots + a_m + a_{m+1}) \\ &\qquad + 2a_2(a_3 + \cdots + a_m + a_{m+1}) + \cdots + 2a_ma_{m+1}, \text{ by (3).}\end{aligned}$$

This result is in accordance with the above law.

Hence, if the law holds for the square of a polynomial of m terms, where m is any positive integer, it also holds for the square of a polynomial of $m + 1$ terms.

But we know that the law holds for the square of a polynomial of three terms, and therefore it holds for the square of a polynomial of four terms; and since it holds for the square of a polynomial of four terms, it also holds for the square of a polynomial of five terms; and so on.

Hence, the law holds for the square of any polynomial.

The above method of proof is known as *Mathematical Induction*.[30]

Example: Expand $\left(2x^2-3x-5\right)^2$.

In accordance with the law, we have

$$\begin{aligned}\left(2x^2-3x-5\right)^2 &= \left(2x^2\right)^2+(-3x)^2+(-5)^2\\ &\quad +2\left(2x^2\right)(-3x)+2\left(2x^2\right)(-5)+2(-3x)(-5)\\ &=4x^4+9x^2+25-12x^3-20x^2+30x\\ &=4x^4-12x^3-11x^2+30x+25.\end{aligned}$$

135. **Cube of a Binomial.**

We find by actual multiplication,

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3, \tag{1}$$
$$(a-b)^3=a^3-3a^2b+3ab^2-b^3. \tag{2}$$

That is,

The cube of the sum of two numbers equals the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

The cube of the difference of two numbers equals the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

1. Find the cube of $a+2b$.
   By (1),

$$\begin{aligned}(a+2b)^3 &= a^3+3a^2(2b)+3a(2b)^2+(2b)^3\\ &= a^3+6a^2b+12ab^2+8b^3.\end{aligned}$$

[30] Mathematical induction is a mathematical proof technique. It is used to prove that a statement P_n holds for every natural number $n=0,1,2,3,\ldots$; that is, the overall statement is a sequence of infinitely many cases P_0, P_1, P_2, P_3, ...

2. Find the cube of $2x^3 - 5y^2$.
   By (2),

$$\begin{aligned}\left(2x^3-5y^2\right)^3 &= \left(2x^3\right)^3 - 3\left(2x^3\right)^2\left(5y^2\right) + 3\left(2x^3\right)\left(5y^2\right)^2 - \left(5y^2\right)^3\\ &= 8x^9 - 60x^6y^2 + 150x^3y^4 - 125y^6.\end{aligned}$$

136. **Cube of a Polynomial.**

By §135, (1),

$$\left(a_1+a_2\right)^3 = a_1^3 + a_2^3 + 3a_1^2a_2 + 3a_1a_2^2. \qquad (1)$$

We also have,

$$\begin{aligned}\left(a_1+a_2+a_3\right)^3 &= \left[\left(a_1+a_2\right)+a_3\right]^3\\ &= \left(a_1+a_2\right)^3 + 3\left(a_1+a_2\right)^2a_3 + 3\left(a_1+a_2\right)a_3^2 + a_3^3\\ &= a_1^3 + 3a_1^2a_2 + 3a_1a_2^2 + a_2^3 + 3a_1^2a_3 + 6a_1a_2a_3\\ &\quad +3a_2^2a_3 + 3a_1a_3^2 + 3a_2a_3^2 + a_3^3\\ &= a_1^3 + a_2^3 + a_3^3 + 3a_1^2a_2 + 3a_1^2a_3 + 3a_2^2a_1 + 3a_2^2a_3\\ &\quad +3a_3^2a_1 + 3a_3^2a_2 + 6a_1a_2a_3 \qquad (2)\end{aligned}$$

The results (1) and (2) are in accordance with the following law:

The cube of a polynomial equals the sum of the cubes of its terms, plus three times the product of the square of each term by each of the other terms, plus six times the product of every three different terms.

We will now prove by *Mathematical Induction* (see §134), that this law holds for the cube of any polynomial.

Assume that the law bolds for the cube of a polynomial of m terms, where m is any positive integer; that is, $\left(a_1+a_2+a_3+\cdots+a_{m-2}+a_{m-1}+a_m\right)^3$

$$\begin{aligned}&= a_1^3 + a_2^3 + \cdots + a_m^3 + 3a_1^2\left(a_2+a_3+\cdots+a_m\right)\\ &\quad + 3a_2^2\left(a_1+a_3+\cdots+a_m\right)\\ &\quad + \cdots + 3a_m^2\left(a_1+a_2+\cdots+a_{m-1}\right)\\ &\quad + 6a_1a_2a_3 + \cdots + 6a_{m-2}a_{m-1}a_m. \qquad (3)\end{aligned}$$

Then, $\left(a_1+a_2+a_3+\cdots+a_{m-1}+a_m+a_{m+1}\right)^3$

$$\begin{aligned}&= \left[\left(a_1+a_2+a_3+\cdots+a_{m-1}+a_m\right)+a_{m+1}\right]^3\\ &= \left(a_1+a_2+a_3+\cdots+a_{m-1}+a_m\right)^3\\ &\quad + 3\left(a_1+a_2+a_3+\cdots+a_{m-1}+a_m\right)^2a_{m+1}\\ &\quad + 3\left(a_1+a_2+a_3+\cdots+a_{m-1}+a_m\right)a_{m+1}^2\\ &\quad + a_{m+1}^3 \text{ (§135).}\end{aligned}$$

Then, by (3) and §134, $\left(a_1+a_2+a_3+\cdots+a_{m-1}+a_m+a_{m+1}\right)^3$

$$\begin{aligned}
= \;& a_1^3+a_2^3+\cdots+a_m^3\\
&+3a_1^2\left(a_2+a_3+\cdots+a_m\right)\\
&+3{a_2}^2\left(a_1+a_3+\cdots+a_m\right)+\ldots\\
&+3{a_m}^2\left(a_1+a_2+\cdots+a_{m-1}\right)\\
&+6a_1a_2a_3+\cdots+6a_{m-2}a_{m-1}a_m\\
&+3a_{m+1}\left(a_1^2+a_2^2+\cdots+a_m^2+2a_1a_2+\cdots+2a_1a_m\right.\\
&\left.+2a_2a_3+\cdots+2a_2a_m+\cdots+2a_{m-1}a_m\right)\\
&+3a_{m+1}^2\left(a_1+a_3+a_3+\cdots+a_m\right)+a_{m+1}^3\\
= \;& a_1^3+a_2^3+\cdots+a_m^3+a_{m+1}^3\\
&+3a_1^2\left(a_2+\cdots+a_m+a_{m+1}\right)\\
&+3a_2^2\left(a_1+a_3+\cdots+a_{m+1}\right)+\ldots\\
&+3a_{m+1}^2\left(a_1+a_2+\cdots+a_m\right)\\
&+6a_1a_2a_3+\cdots+6a_{m-1}a_ma_{m+1}.
\end{aligned}$$

This result is in accordance with the above law.

Hence, if the law holds for the cube of a polynomial of m terms, where m is any positive integet, it also holds for the cube of a polynomial of $m+1$ terms.

But we know that the law holds for the cube of a polynomial of three terms, and therefore it holds for the cube of a polynomial of four terms; and since it holds for the cube of, a polynomial of four terms, it also holds for the cube of a polynomial of five terms; and so on.

Hence, the law holds for the cube of any polynomial.

Example: Expand $\left(2x^3-x^2+2x-3\right)^3$.

In accordance with the above law, we have

$$\begin{aligned}
&\left(2x^3-x^2+2x-3\right)^3\\
&=\left(2x^3\right)^3+\left(-x^2\right)^3+(2x)^3+(-3)^3+3\left(2x^3\right)^2\left(-x^2+2x-3\right)\\
&\quad+3\left(-x^2\right)^2\left(2x^3+2x-3\right)+3(2x)^2\left(2x^3-x^2-3\right)\\
&\quad+3(-3)^2\left(2x^3-x^2+2x\right)\\
&\quad+6\left(2x^3\right)\left(-x^2\right)(2x)+6\left(2x^3\right)\left(-x^2\right)(-3)\\
&\quad+6\left(2x^3\right)(2x)(-3)+6\left(-x^2\right)(2x)(-3)
\end{aligned}$$

Expanding and simplifying, $\left(2x^3-x^2+2x-3\right)^3$

$$=8x^9-12x^8+30x^7-61x^6+66x^5-93x^4+98x^3-63x^2+54x-27.$$

6.2 Examples

> "Pure mathematics is the world's best game. It is more absorbing than chess, more of a gamble than poker, and lasts longer than Monopoly. It's free. It can be played anywhere—Archimedes did it in a bathtub."
>
> —Richard J. Trudeau

Seeing an example before starting the exercises may be a needed gateway to success. However, you may also contact Ron Bannon ([redacted]) and request one-on-one assistance.

1. Square $3a^2 - 2b$.

Solution:

$$\begin{aligned} (A-B)^2 &= A^2 - 2AB + B^2 \\ \left(3a^2 - 2b\right)^2 &= \left(3a^2\right)^2 - 2\left(3a^2\right)(2b) + (2b)^2 \\ &= 9a^4 - 12a^2b + 4b^2 \end{aligned}$$

2. Square $-4x^3 + 9$.

Solution:

$$\begin{aligned} (A+B)^2 &= A^2 + 2AB + B^2 \\ \left(-4x^3 + 9\right)^2 &= \left(-4x^3\right)^2 + 2\left(-4x^3\right)(9) + (9)^2 \\ &= 16x^6 - 72x^3 + 81 \end{aligned}$$

3. Multiply $6a + 5b^3$ by $6a - 5b^3$.

Solution:

$$\begin{aligned} (A+B)(A-B) &= A^2 - B^2 \\ \left(6a + 5b^3\right)\left(6a - 5b^3\right) &= (6a)^2 - \left(5b^3\right)^2 \\ &= 36a^2 - 25b^6 \end{aligned}$$

4. Multiply $-x^2+4$ by $-x^2-4$.

Solution:

$$\begin{aligned}(A+B)(A-B) &= A^2-B^2\\ \left(-x^2+4\right)\left(-x^2-4\right) &= \left(-x^2\right)^2-(4)^2\\ &= x^4-16\end{aligned}$$

5. Expand $(a+b-c)(a-b+c)$.

Solution:

$$\begin{aligned}(a+b-c)(a-b+c) &= (a+[b-c])(a-[b-c])\\ &= a^2-[b-c]^2\\ &= a^2-\left(b^2-2bc+c^2\right)\\ &= a^2-b^2-c^2+2bc\end{aligned}$$

6. Expand $(x+y+z)(x-y+z)$.

Solution:

$$\begin{aligned}(x+y+z)(x-y+z) &= ([x+z]+y)([x+z]-y)\\ &= [x+z]^2-y^2\\ &= x^2+2xz+z^2-y^2\end{aligned}$$

7. Multiply $x-5$ by $x+3$.

Solution:

$$\begin{aligned}(x+A)(x+B) &= x^2+(A+B)x+AB\\ (x-5)(x+3) &= x^2-2x-15\end{aligned}$$

8. Multiply $ab-4$ by $ab+7$.

Solution:

$$\begin{aligned}(x+A)(x+B) &= x^2+(A+B)x+AB\\ (ab-4)(ab+7) &= a^2b^2+3ab-28\end{aligned}$$

9. Multiply $m+n+6$ by $m+n+8$.

Solution:

$$\begin{aligned}([m+n]+6)([m+n]+8) &= [m+n]^2+14[m+n]+48\\ &= m^2+2mn+n^2+14m+14n+48\end{aligned}$$

10. Expand $\left(2x^2-3x-5\right)^2$.

Solution:

$$\begin{aligned}\left(2x^2-3x-5\right)^2 &= \left(2x^2-3x\right)^2-2\cdot 5\left(2x^2-3x\right)+(5)^2\\ &= 4x^4-12x^3-11x^2+30x+25\end{aligned}$$

11. Find the cube of $a+2b$.

Solution:

$$\begin{aligned}(A+B)^3 &= A^3+3A^2B+3AB^2+B^3\\ (a+2b)^3 &= (a)^3+3(a)^2(2b)+3(a)(2b)^2+(2b)^3\\ &= a^3+6a^2b+12ab^2+8b^3\end{aligned}$$

12. Find the cube of $2x^3-5y^2$.

Solution:

$$\begin{aligned}(A-B)^3 &= A^3-3A^2B+3AB^2-B^3\\ \left(2x^3-5y^2\right)^3 &= \left(2x^3\right)^3-3\left(2x^3\right)^2\left(5y^2\right)+3\left(2x^3\right)\left(5y^2\right)^2-\left(5y^2\right)^3\\ &= 8x^9-60x^6y^2+150x^3y^4-125y^6\end{aligned}$$

13. Expand $\left(2x^3 - x^2 + 2x - 3\right)^3$.

Solution:

$$\begin{aligned}
\left(2x^3 - x^2 + 2x - 3\right)^3 &= \left(\left[2x^3 - x^2\right] + \left[2x - 3\right]\right)^3 \\
&= \left[2x^3 - x^2\right]^3 + 3\left[2x^3 - x^2\right]^2\left[2x - 3\right] \\
&\qquad + 3\left[2x^3 - x^2\right]\left[2x - 3\right]^2 + \left[2x - 3\right]^3 \\
&\vdots \quad \text{Very tedious} \ldots \\
&= 8x^9 - 12x^8 + 30x^7 - 61x^6 + 66x^5 - 93x^4 + 98x^3 \\
&\qquad - 63x^2 + 54x - 27
\end{aligned}$$

6.3 Exercises

Although some problems are very challenging, you should be able to do most of the following exercises. Solutions and semi-detailed work appear after each problem as a way to validate your progress. Updates to this document will undoubtedly happen, and I encourage feedback[31] related to improving this document to help me improve the content. Work provided may not be complete, but it should be sufficient to understand the provided solutions.

Although I appreciate what Wells does in his text, I may be taking a different approach to solving *some* of the problems. Using your head and having options is needed if you plan to move forward in mathematics.

If, for whatever reason, you need me to show additional work, please do contact me, and I will make every effort to expand the provided work. You're not alone if you can't follow a particular solution. However, it would be best to have a good grasp of algebra to solve many of these problems. Wells' exercises follow directly from his lessons—that is, if you understand the examples given before starting the exercises, you should be able to work the entire problem set. Mistakes happen, so please do ask if you think something is amiss.

One final note, although my solutions may not be the *most elegant*, they are nonetheless the way I *initially* did the problem. Bear in mind that I *welcome more elegant* solutions and will certainly include them in future versions of this document, with *attributions*, of course!

1. Simplify $\left(6a^3x^2\right)^3$ by inspection.

Solution:

$$\left(6a^3x^2\right)^3 = 216a^9x^6$$

2. Simplify $\left(-4ab^5c^4\right)^4$ by inspection.

Solution:

$$\left(-4ab^5c^4\right)^4 = 256a^4b^{20}c^{16}$$

3. Simplify $\left(-3x^3yz^2\right)^5$ by inspection.

Solution:

$$\left(-3x^3yz^2\right)^5 = -243x^{15}y^5z^{10}$$

[31] Contact Ron Bannon at [redacted] if you have questions or concerns related to these problems. Try your best to indicate the source of your concern by *citing* the section and problem number.

4. Simplify $\left(3+7x^2\right)^2$ by inspection.

Solution:

$$\left(3+7x^2\right)^2 = 9+42x^2+49x^4$$

5. Simplify $\left(2a^3-5b^2c\right)^2$ by inspection.

Solution:

$$\left(2a^3-5b^2c\right)^2 = 4a^6-20a^3b^2c+25b^4c^2$$

6. Simplify $\left(-m^4n^6+4p^5\right)^2$ by inspection.

Solution:

$$\left(-m^4n^6+4p^5\right)^2 = m^8n^{12}-8m^4n^6p^5+16p^{10}$$

7. Simplify $(-6xy-11xz)^2$ by inspection.

Solution:

$$(-6xy-11xz)^2 = 36x^2y^2+132x^2yz+121x^2z^2$$

8. Simplify $(8x^p-9x^q)^2$ by inspection; p and q being positive integers.

Solution:

$$(8x^p-9x^q)^2 = 64x^{2p}-144x^px^q+81x^{2q}$$

9. Simplify $\left(5a^2+12b^3c\right)\left(5a^2-12b^3c\right)$ by inspection.

Solution:

$$\left(5a^2+12b^3c\right)\left(5a^2-12b^3c\right) = 25a^4-144b^6c^2$$

10. Simplify $\left(-10m^4n+13x^5\right)\left(-10m^4n-13x^5\right)$ by inspection.

Solution:

$$\left(-10m^4n+13x^5\right)\left(-10m^4n-13x^5\right) = 100m^8n^2-169x^{10}$$

11. Simplify $\left(a^{2m}+x^{3n}\right)\left(a^{2m}-x^{3n}\right)$ by inspection; m and n being positive integers.

Solution:

$$\left(a^{2m}+x^{3n}\right)\left(a^{2m}-x^{3n}\right) = a^{4m}-x^{6n}$$

12. Expand $(a-b+c)(a-b-c)$.

Solution:

$$(a-b+c)(a-b-c) = a^2-2ab+b^2-c^2$$

13. Expand $(x+y+3)(x-y-3)$.

Solution:

$$(x+y+3)(x-y-3) = x^2-y^2-6y-9$$

14. Expand $\left(x^2+xy+y^2\right)\left(x^2-xy+y^2\right)$.

Solution:

$$\left(x^2+xy+y^2\right)\left(x^2-xy+y^2\right) = x^4+x^2y^2+y^4$$

15. Expand $\left(a^2+5a-4\right)\left(a^2-5a+4\right)$.

Solution:

$$\left(a^2+5a-4\right)\left(a^2-5a+4\right) = a^4-25a^2+40a-16$$

16. Expand $\left(4x^2+3x+7\right)\left(4x^2+3x-7\right)$.

Solution:

$$\left(4x^2+3x+7\right)\left(4x^2+3x-7\right) = 16x^4+24x^3+9x^2-49$$

17. Expand $\left(m^4+5m^2n^2+2n^4\right)\left(m^4-5m^2n^2-2n^4\right)$.

Solution:

$$= m^8-25m^4n^4-20m^2n^6-4n^8$$

18. Expand $(x+2)(x+10)$.

Solution:

$$(x+2)(x+10) = x^2+12x+20$$

19. Expand $(x-5)(x+7)$.

Solution:

$$(x-5)(x+7) \quad = \quad x^2+2x-35$$

20. Expand $\left(x^2-4\right)\left(x^2-14\right)$.

Solution:

$$\left(x^2-4\right)\left(x^2-14\right) \quad = \quad x^4-18x^2+56$$

21. Expand $(x+7a)(x-15a)$.

Solution:

$$(x+7a)(x-15a) \quad = \quad x^2-8ax-105a^2$$

22. Expand $(mn+11)(mn+2)$.

Solution:

$$(mn+11)(mn+2) \quad = \quad m^2n^2+13mn+22$$

23. Expand $\left(a^2b+3c^3\right)\left(a^2b-8c^3\right)$.

Solution:

$$\left(a^2b+3c^3\right)\left(a^2b-8c^3\right) \quad = \quad a^4b^2-8a^2bc^3+3a^2bc^3-24c^6$$

24. Expand $(a-b-5)(a-b-9)$.

Solution:

$$(a-b-5)(a-b-9) = a^2-2ab+b^2-14a+14b+45$$

25. Expand $\left(x+y-6z^2\right)\left(x+y+12z^2\right)$.

Solution:

$$\left(x+y-6z^2\right)\left(x+y+12z^2\right) = -72z^4+6xz^2+6yz^2+x^2+2xy+y^2$$

26. Expand $\left(3x^2+5x-4\right)^2$.

Solution:

$$\left(3x^2+5x-4\right)^2 = 9x^4+30x^3+x^2-40x+16$$

27. Expand $(a-b-c+d)^2$.

Solution:

$$(a-b-c+d)^2 = a^2-2ab+b^2-2ac+2bc+c^2+2ad-2bd-2cd+d^2$$

28. Expand $\left(2x^3-3x^2+x-2\right)^2$.

Solution:

$$\left(2x^3-3x^2+x-2\right)^2 = 4x^6-12x^5+13x^4-14x^3+13x^2-4x+4$$

29. Expand $\left(a^4+3a^3-4a^2-2a+1\right)^2$.

Solution:

$$\left(a^4+3a^3-4a^2-2a+1\right)^2 = a^8+6a^7+a^6-28a^5+6a^4+22a^3-4a^2-4a+1$$

30. Expand $(a+3b)^3$ by inspection.

Solution:

$$(a+3b)^3 = a^3+9a^2b+27ab^2+27b^3$$

31. Expand $\left(7x^4-x^3\right)^3$ by inspection.

Solution:

$$\left(7x^4-x^3\right)^3 = 343x^{12}-147x^{11}+21x^{10}-x^9$$

32. Expand $\left(3a^2b+2c^3\right)^3$ by inspection.

Solution:

$$\left(3a^2b+2c^3\right)^3 = 27a^6b^3+54a^4b^2c^3+36a^2bc^6+8c^9$$

33. Expand $\left(5mx^2-4ny^3\right)^3$ by inspection.

Solution:

$$\left(5mx^2-4ny^3\right)^3 = -64n^3y^9+240mn^2x^2y^6-300m^2nx^4y^3+125m^3x^6$$

34. Expand $\left(a^2+a-2\right)^3$.

Solution:

$$\left(a^2+a-2\right)^3 = a^6+3a^5-3a^4-11a^3+6a^2+12a-8$$

35. Expand $\left(2x^2-4x+3\right)^3$.

Solution:

$$\left(2x^2-4x+3\right)^3 = 8x^6-48x^5+132x^4-208x^3+198x^2-108x+27$$

36. Expand $(a-b+c-d)^3$.

Solution:

$$\begin{aligned} &= a^3-3a^2b+3ab^2-b^3+3a^2c-6abc+3b^2c+3ac^2-3bc^2+c^3-3a^2d \\ &\quad +6abd-3b^2d-6acd+6bcd-3c^2d+3ad^2-3bd^2+3cd^2-d^3 \end{aligned}$$

37. Expand $\left(3x^3-4x^2-2x+1\right)^3$.

Solution:

$$= 27x^9-108x^8+90x^7+107x^6-132x^5-36x^4+49x^3-6x+1$$

38. Simplify $\left(3a^2+5b\right)^2\left(3a^2-5b\right)^2$.

Solution:

$$\left(3a^2+5b\right)^2\left(3a^2-5b\right)^2 = 81a^8-450a^4b^2+625b^4$$

39. Simplify $(x+5)(x-2)(x-5)(x+2)$.

Solution:

$$(x+5)(x-2)(x-5)(x+2) = x^4 - 29x^2 + 100$$

40. Simplify $(2-x)(2+x)\left(4+x^2\right)$.

Solution:

$$(2-x)(2+x)\left(4+x^2\right) = 16 - x^4$$

41. Simplify $(x+1)^3(x-1)^3$.

Solution:

$$(x+1)^3(x-1)^3 = x^6 - 3x^4 + 3x^2 - 1$$

42. Simplify $(x+y-z)^2(x-y+z)^2$.

Solution:

$$= x^4 - 2x^2y^2 + y^4 + 4x^2yz - 4y^3z - 2x^2z^2 + 6y^2z^2 - 4yz^3 + z^4$$

43. Simplify $(a+b+c)^3(a+b-c)^3$.

Solution:

$$\begin{aligned} = \; & a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 - 3a^4c^2 - 12a^3bc^2 \\ & - 18a^2b^2c^2 - 12ab^3c^2 - 3b^4c^2 + 3a^2c^4 + 6abc^4 + 3b^2c^4 - c^6 \end{aligned}$$

44. Simplify $(x+y+z)^2+(y+z-x)^2+(z+x-y)^2+(x+y-z)^2$.

Solution:

$$(x+y+z)^2+(y+z-x)^2+(z+x-y)^2+(x+y-z)^2 = 4x^2+4y^2+4z^2$$

45. Simplify $(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.

Solution:

$$= -a^4+2a^2b^2-b^4+2a^2c^2+2b^2c^2-c^4$$

46. Simplify $(m+n)^3-(m-n)^3-3(m+n)^2(m-n)+3(m+n)(m-n)^2$.

Solution:

$$(m+n)^3-(m-n)^3-3(m+n)^2(m-n)+3(m+n)(m-n)^2 = 8n^3$$

7.1 Wells' Chapter VII

137. We find by actual division,

$$\frac{a^2-b^2}{a+b} = a-b. \quad (1)$$
$$\frac{a^2-b^2}{a-b} = a+b. \quad (2)$$

That is,

If the difference of the squares of two numbers be divided by the sum of the numbers, the quotient is the difference of the numbers.

If the difference of the squares of two numbers be divided by the difference of the numbers, the quotient is the sum of the numbers.

Example: Divide $25y^2z^4-9$ by $5yz^2+3$.

By §130, $25y^2z^4$ is the square of $5yz^2$'

Then, by (1),

$$\frac{25y^2z^4-9}{5yz^2+3} = 5yz^2-3.$$

138. We find by actual division,

$$\frac{a^3+b^3}{a+b} = a^2-ab+b^2. \quad (1)$$
$$\frac{a^3-b^3}{a-b} = a^2+ab+b^2. \quad (2)$$

That is,

If the sum of the cubes of two numbers be divided by the sum of the numbers, the quotient is the square of the first number, minus the product of the first by the second, plus the square of the second number.

If the difference of the cubes of two numbers be divided by the difference of the numbers, the quotient is the square of the first number, plus the product of the first by the second, plus the square of the second number.

Example: Divide $27a^3-b^3$ by $3a-b$.

By §130, $27a^3$ is the cube of $3a$.

Then, by (2),

$$\frac{27a^3-b^3}{3a-b} = 9a^2+3ab+b^2.$$

139. **The Remainder Theorem.**

Let it be required to divide $2x^3 - 7x^2 + 10x - 3$ by $x - 2$.

$$
\begin{array}{r|l}
 & 2x^2 \quad - 3x + 4 \\
\hline
x - 2 & 2x^3 - 7x^2 + 10x - 3 \\
 & -2x^3 + 4x^2 \\
\cline{2-2}
 & \quad - 3x^2 + 10x \\
 & \quad 3x^2 \quad - 6x \\
\cline{2-2}
 & \qquad 4x - 3 \\
 & \qquad - 4x + 8 \\
\cline{2-2}
 & \qquad\qquad 5
\end{array}
$$

The division is not exact, and there is a final remainder 5 .

Now if we substitute 2 for x in the dividend, we have, $2 \times 2^3 - 7 \times 2^2 + 10 \times 2 - 3$, which equals 5.

This exemplifies the following law:

If any rational integral polynomial, involving x, be not divisible by $x - a$, the remainder of the division equals the result obtained by substituting a for x in the given polynomial.

The above is called *The Remainder Theorem*.

To prove the theorem, let D be any rational integral polynomial, involving x, not divisible by $x - a$.

Let the division be carried out until a remainder is obtained which does not contain x.

Let Q denote the quotient, and R the remainder.

Since the dividend equals the product of the divisor and quotient, plus the remainder, we have, $Q(x - a) + R = D$.

Substitute in this equation a for x.

The term $Q(x - a)$ becomes zero; and since R does not contain x, it is not changed, whatever value is given to x.

Then, R must equal the result obtained by substituting a for x in D.

140. **The Factor Theorem.**

If any rational integral polynomial, involving x, becomes zero when x is put equal to a, the polynomial has $x - a$ as a factor.

For by §139, the remainder obtained by dividing the polynomial by $x - a$ is zero.

141. It follows from §140 that,

If any rational integral polynomial, involving x, becomes zero when x is put equal to $-a$, the polynomial has $x + a$ as a factor.

142. We will now prove that, if n is any positive integer,

I. $a^n - b^n$ is always divisible by $a - b$.

II. $a^n - b^n$ is divisible by $a + b$ if n is even.

III. $a^n + b^n$ is divisible by $a + b$ if n is odd.

IV. $a^n + b^n$ is divisible by neither $a + b$ nor $a - b$ if n is even.

Proof of I. If b be substituted for a in $a^n - b^n$, the result is $b^n - b^n$, or 0.

Then by §140, $a^n - b^n$ has $a - b$ as a factor.

Proof of II. If $-b$ be substituted for a in $a^x - b^n$, the result is $(-b)^n - b^n$; or, since n is even, $b^x - b^n$, or 0.

Then by §141, $a^n - b^n$ has $a + b$ as a factor.

Proof of III. If $-b$ be substituted for a in $a^n + b^n$, the result is $(-b)^n + b^n$; or, since n is odd, $-b^n + b^n$, or 0.

Then, $a^n + b^n$ has $a + b$ as a factor.

Proof of IV. If $-b$ or $+b$ be substituted for a in $a^n + b^n$, the results are $(-b)^n + b^n$ or $b^n + b^n$, respectively.

Since n is even, neither of these is zero.

Then, neither $a + b$ nor $a - b$ is a factor of $a^n + b^n$.

143. We find by actual division

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2 b + ab^2 - b^3,$$
$$\frac{a^4 - b^4}{a - b} = a^3 + a^2 b + ab^2 + b^3,$$
$$\frac{a^5 + b^5}{a + b} = a^4 - a^3 b + a^2 b^2 - ab^3 + b^4,$$
$$\frac{a^5 - b^5}{a - b} = a^4 + a^3 b + a^2 b^2 + ab^3 + b^4; \text{ etc.}$$

In these results, we observe the following laws:

I. *The exponent of a in the first term of the quotient is less by 1 than its exponent in the dividend, and decreases by 1 in each succeeding term.*

II. *The exponent of b in the second term of the quotient is 1, and increases by 1 in each succeeding term.*

III. *If the divisor is $a - b$, all the terms of the quotient are positive; if the divisor is $a + b$, the terms of the quotient are alternately positive and negative.*

144. We will now prove, by *Mathematical Induction*, that the laws of §143 hold universally. Assume the laws to hold for $\dfrac{a^n-b^n}{a-b}$, where n is any positive integer. Then,

$$\frac{a^n-b^n}{a-b} = a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+b^{n-1}. \quad (1)$$

$$\begin{aligned}\text{Now, } \frac{a^{n+1}-b^{n+1}}{a-b} &= \frac{a^{n+1}-a^nb+a^nb-b^{n+1}}{a-b} \\ &= \frac{a^n(a-b)+b\left(a^n-b^n\right)}{a-b} \\ &= a^n+b\left(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+b^{n-1}\right), \text{ by (1)} \\ &= a^n+a^{n-1}b+a^{n-2}b^2+\cdots+b^n.\end{aligned}$$

This result is in accordance with the laws of §143.

Hence, if the laws hold for the quotient of the difference of two like powers of a and b divided by $a-b$, they also hold for the quotient of the difference of the next higher powers of a and b divided by $a-b$.

But we know that they hold for $\dfrac{a^5-b^5}{a-b}$, and therefore they hold for $\dfrac{a^6-b^6}{a-b}$; and since they hold for $\dfrac{a^6-b^6}{a-b}$, they hold for $\dfrac{a^7-b^7}{a-b}$; and so on.

Hence, the laws hold for $\dfrac{a^n-b^n}{a-b}$, where n is any positive integer.

Putting $-b$ for b in (1), we have

$$\frac{a^n-(-b)^n}{a-(-b)} = a^{n-1}+a^{n-2}(-b)+\cdots+(-b)^{n-1}$$

If n is even, $(-b)^n=b^n$, and $(-b)^{n-1}=-b^{n-1}$ (§130).

Whence,

$$\frac{a^n-b^n}{a+b} = a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots-b^{n-1}. \quad (2)$$

If n is odd, $(-b)^n=-b^n$, and $(-b)^{n-1}=+b^{n-1}$.

Whence,

$$\frac{a^n+b^n}{a+b} = a^{n-1}-a^{n-2}b+a^{n-3}b^2-\cdots+b^{n-1}. \quad (3)$$

Equations (2) and (3) are in accordance with the laws of §143.

Hence, the laws hold for $\dfrac{a^n-b^n}{a+b}$, where n is any even positive integer, and for $\dfrac{a^n+b^n}{a+b}$, where n is any odd positive integer.

145. 1. Divide $a^7 - b^7$ by $a - b$.

By §143, $\dfrac{a^7 - b^7}{a - b} = a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6$.

2. Divide $16x^4 - 81$ by $2x + 3$

By §130, $16x^4 = (2x)^4$. Then,

$$\begin{aligned}\frac{16x^4 - 81}{2x + 3} &= (2x)^3 - (2x)^2 \cdot 3 + (2x) \cdot 3^2 - 3^3\\ &= 8x^3 - 12x^2 + 18x - 27\end{aligned}$$

The absolute value of any term after the first, in equations (1), (2), and (3), of §144, may be obtained by dividing the absolute value of the preceding term by a, and multiplying the result by b.

This would be the shortest method if the numbers involved were large.

7.2 Examples

> "There are two ways to do great mathematics. The first is to be smarter than everybody else. The second way is to be stupider than everybody else—but persistent."
>
> —Raoul Bott

Seeing an example before starting the exercises may be a needed gateway to success. However, you may also contact Ron Bannon ([redacted]) and request one-on-one assistance.

These patterns should be rcognized:

$$\begin{aligned}
\frac{A^2-B^2}{A+B} &= \frac{\cancel{(A+B)}(A-B)}{\cancel{(A+B)}} = A-B,\\
\frac{A^2-B^2}{A-B} &= \frac{\cancel{(A-B)}(A+B)}{\cancel{(A-B)}} = A+B,\\
\frac{A^3+B^3}{A+B} &= \frac{\cancel{(A+B)}\left(A^2-AB+B^2\right)}{\cancel{(A+B)}} = A^2-AB+B^2,\\
\frac{A^3-B^3}{A-B} &= \frac{\cancel{(A-B)}\left(A^2+AB+B^2\right)}{\cancel{(A-B)}} = A^2+AB+B^2.
\end{aligned}$$

1. Divide $27a^3-b^3$ by $3a-b$.

Solution: Here $A=3a$ and $B=b$

$$\begin{aligned}
\frac{A^3-B^3}{A-B} &= A^2+AB+B^2\\
\frac{27a^3-b^3}{3a-b} &= 9a^2+3ab+b^2.
\end{aligned}$$

2. Long divide $2x^3-7x^2+10x-3$ by $x-2$.

Solution:

$$\begin{array}{r|rrrr}
 & 2x^2 & -3x & +4 & \\
\hline
x-2 & 2x^3 & -7x^2 & +10x & -3\\
 & -2x^3 & +4x^2 & & \\
\hline
 & & -3x^2 & +10x & \\
 & & 3x^2 & -6x & \\
\hline
 & & & 4x & -3\\
 & & & -4x & +8\\
\hline
 & & & & 5
\end{array}$$

$$\frac{2x^3 - 7x^2 + 10x - 3}{x - 2} \quad = \quad 2x^2 - 3x + 4 + \frac{5}{x - 2}$$

Here you should note that

$$2x^3 - 7x^2 + 10x - 3 \quad = \quad \left(2x^2 - 3x + 4\right)(x - 2) + 5.$$

If you now substitute $x = 2$ you'll get

$$2\,(2)^3 - 7\,(2)^2 + 10\,(2) - 3 \quad = \quad 0 + 5.$$

This illustrates what is referred to as The Remainder Theorem. That is, if any rational integral polynomial, involving x, be not divisible by $x - a$, the remainder of the division equals the result obtained by substituting a for x in the given polynomial.

From this follows The Factor Theorem. If any rational integral polynomial, involving x, becomes zero when x is put equal to a, the polynomial has $x - a$ as a factor.

Using The Factor Theorem, we can prove that, if n is any positive integer

(a) $a^n - b^n$ is always divisible by $a - b$. Proof: Using The Factor Theorem, letting $a = b$ into $a^n - b^n = 0$.

(b) $a^n - b^n$ is divisible by $a + b$ if n is even. Proof: Using The Factor Theorem, letting $a = -b$ into $a^n - b^n = 0$ iff n is even. If n were odd you'd get $-2b^n$, which is zero iff $b = 0$.

(c) $a^n + b^n$ is divisible by $a + b$ if n is odd. Proof: Using The Factor Theorem, letting $a = -b$ into $a^n + b^n = 0$ iff n is odd. If n were even you'd get $2b^n$, which is zero iff $b = 0$.

(d) $a^n + b^n$ is divisible by neither $a + b$ nor $a - b$ if n is even. Proof: Using The Factor Theorem, letting $a = \pm b$ into $a^n + b^n$ if n is even gives $2b^2$ which is zero iff $b = 0$.

3. Long divide: $a^4 - b^4$ by $a + b$; $a^4 - b^4$ by $a - b$; $a^5 - b^5$ by $a + b$; $a^5 - b^5$ by $a - b$;

Solution:

$$\begin{aligned} \frac{a^4 - b^4}{a + b} &= a^3 - a^2 b + ab^2 - b^3 \\ \frac{a^4 - b^4}{a - b} &= a^3 + a^2 b + ab^2 + b^3 \\ \frac{a^5 - b^5}{a + b} &= a^4 - a^3 b + a^2 b^2 - ab^3 + b^4 \\ \frac{a^5 - b^5}{a - b} &= a^4 + a^3 b + a^2 b^2 + ab^3 + b^4 \end{aligned}$$

We have a general pattern that can be proved by Mathematical Induction. Here n is a positive integer.

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \cdots + a^2 b^{n-3} + ab^{n-2} + b^{n-1}$$

Now replace b by $-b$ and you'll get:

$$= \frac{a^n - (-b)^n}{a - (-b)}$$

If n is even $(-b)^n = b^n$:

$$\frac{a^n - b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \cdots - a^2 b^{n-3} + ab^{n-2} - b^{n-1};$$

and if n is odd $(-b)^n = -b^n$:

$$\frac{a^n + b^n}{a + b} = a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3 + \cdots - a^2 b^{n-3} + ab^{n-2} - b^{n-1}.$$

4. Divide $a^7 - b^7$ by $a - b$,

Solution:

$$\frac{a^7 - b^7}{a - b} = a^6 + a^5 b + a^4 b^2 + a^3 b^3 + a^2 b^4 + ab^5 + b^6$$

5. Divide $16x^4 - 81$ by $2x + 3$,

Solution:

$$\begin{aligned} \frac{16x^4 - 81}{2x + 3} &= (2x)^3 - (2x)^2 \cdot 3 + (2x) \cdot 3^2 - 3^3 \\ &= 8x^3 - 12x^2 + 18x - 27 \end{aligned}$$

7.3 Exercises

Although some problems are very challenging, you should be able to do most of the following exercises. Solutions and semi-detailed work appear after each problem as a way to validate your progress. Updates to this document will undoubtedly happen, and I encourage feedback[32] related to improving this document to help me improve the content. Work provided may not be complete, but it should be sufficient to understand the provided solutions.

Although I appreciate what Wells does in his text, I may be taking a different approach to solving *some* of the problems. Using your head and having options is needed if you plan to move forward in mathematics.

If, for whatever reason, you need me to show additional work, please do contact me, and I will make every effort to expand the provided work. You're not alone if you can't follow a particular solution. However, it would be best to have a good grasp of algebra to solve many of these problems. Wells' exercises follow directly from his lessons—that is, if you understand the examples given before starting the exercises, you should be able to work the entire problem set. Mistakes happen, so please do ask if you think something is amiss.

One final note, although my solutions may not be the *most elegant*, they are nonetheless the way I *initially* did the problem. Bear in mind that I *welcome more elegant* solutions and will certainly include them in future versions of this document, with *attributions*, of course!

1. Write by inspection the value of:

$$\frac{36a^2-49}{6a+7}.$$

Solution: By inspection means that the middle part would be done in your head and not written down. You may in time be able to do many steps in your head, but if you're making mistakes I suggest you write down the steps!

$$\frac{36a^2-49}{6a+7} = \frac{(6a-7)\cancel{(6a+7)}}{\cancel{(6a+7)}} = 6a-7$$

2. Write by inspection the value of:

$$\frac{121x^6-64y^4z^2}{11x^3-8y^2z}.$$

[32] Contact Ron Bannon at [redacted] if you have questions or concerns related to these problems. Try your best to indicate the source of your concern by *citing* the section and problem number.

Solution:

$$\frac{121x^6 - 64y^4z^2}{11x^3 - 8y^2z} = \frac{\cancel{\left(11x^3 - 8y^2z\right)}\left(11x^3 + 8y^2z\right)}{\cancel{\left(11x^3 - 8y^2z\right)}} = 11x^3 + 8y^2z$$

3. Write by inspection the value of:

$$\frac{n^3 - 1}{n - 1}.$$

Solution:

$$\frac{n^3 - 1}{n - 1} = \frac{\cancel{(n - 1)}\left(n^2 + n + 1\right)}{\cancel{(n - 1)}} = n^2 + n + 1$$

4. Write by inspection the value of:

$$\frac{8 + m^6}{2 + m^2}.$$

Solution:

$$\begin{aligned} \frac{8 + m^6}{2 + m^2} &= \frac{\cancel{\left(2 + m^2\right)}\left(4 - 2m^2 + m^4\right)}{\cancel{\left(2 + m^2\right)}} = 4 - 2m^2 + m^4 \\ &= m^4 - 2m^2 + 4 \end{aligned}$$

5. Write by inspection the value of:

$$\frac{125a^6 - 27x^3}{5a^2 - 3x}.$$

Solution:

$$\frac{125a^6-27x^3}{5a^2-3x} = \frac{\cancel{\left(a^2-3x\right)}\left(25a^4+15a^2x+9x^2\right)}{\cancel{\left(a^2-3x\right)}} = 25a^4+15a^2x+9x^2$$

6. Write by inspection the value of:

$$\frac{216m^3n^6+343p^9}{6mn^2+7p^3}.$$

Solution:

$$\begin{aligned}\frac{216m^3n^6+343p^9}{6mn^2+7p^3} &= \frac{\cancel{\left(6mn^2+7p^3\right)}\left(36m^2n^4-42mn^2p^3+49p^6\right)}{\cancel{\left(6mn^2+7p^3\right)}}\\ &= 36m^2n^4-42mn^2p^3+49p^6\end{aligned}$$

7. Write by inspection the value of:

$$\frac{a^4-b^4}{a+b}.$$

Solution:

$$\begin{aligned}\frac{a^4-b^4}{a+b} &= \frac{\left(a^2-b^2\right)\left(a^2+b^2\right)}{a+b}\\ &= \frac{(a-b)(a+b)\left(a^2+b^2\right)}{a+b}\\ &= (a-b)\left(a^2+b^2\right)\\ &= a^3-a^2b+ab^2-b^3\end{aligned}$$

8. Write by inspection the value of:

$$\frac{m^5-n^5}{m-n}.$$

Solution:

$$\frac{m^5 - n^5}{m - n} = m^4 + m^3 n + m^2 n^2 + mn^3 + n^4$$

9. Write by inspection the value of:

$$\frac{1 - x^6}{1 - x}.$$

Solution:

$$\begin{aligned} \frac{1 - x^6}{1 - x} &= (1 + x)\left(1 + x^2 + x^4\right) \\ &= 1 + x + x^2 + x^3 + x^4 + x^5 \end{aligned}$$

10. Write by inspection the value of:

$$\frac{a^7 + x^7}{a + x}.$$

Solution:

$$\frac{a^7 + x^7}{a + x} = a^6 - a^5 x + a^4 x^2 - a^3 x^3 + a^2 x^4 - a x^5 + x^6$$

11. Write by inspection the value of:

$$\frac{n^9 - x^9}{n - x}.$$

Solution:

$$\begin{aligned} \frac{n^9 - x^9}{n - x} &= \left(n^2 + nx + x^2\right)\left(n^6 + n^3 x^3 + x^6\right) \\ &= n^8 + n^7 x + n^6 x^2 + n^5 x^3 + n^4 x^4 + n^3 x^5 + n^2 x^6 + n x^7 + x^8 \end{aligned}$$

12. Write by inspection the value of:

$$\frac{a^6 - 64b^6}{a - 2b}.$$

Solution:

$$\frac{a^6 - 64b^6}{a - 2b} = a^5 + 2a^4b + 4a^3b^2 + 8a^2b^3 + 16ab^4 + 32b^5$$

13. Write by inspection the value of:

$$\frac{625m^4 - 256}{5m - 4}.$$

Solution:

$$\begin{aligned}\frac{625m^4 - 256}{5m - 4} &= (5m+4)\left(25m^2+16\right)\\ &= 125m^3 + 100m^2 + 80m + 64\end{aligned}$$

14. Write by inspection the value of:

$$\frac{256a^8 - x^8}{2a + x}.$$

Solution:

$$\begin{aligned}\frac{256a^8 - x^8}{2a + x} &= (2a-x)\left(4a^2+x^2\right)\left(16a^4+x^4\right)\\ &= 128a^7 - 64a^6x + 32a^5x^2 - 16a^4x^3 + 8a^3x^4 - 4a^2x^5 + 2ax^6 - x^7\end{aligned}$$

15. Write by inspection the value of:

$$\frac{243x^5 + 1024y^5}{3x + 4y}.$$

Solution:

$$\frac{243x^5 + 1024y^5}{3x + 4y} = 81x^4 - 108x^3y + 144x^2y^2 - 192xy^3 + 256y^4$$

8.1 Wells' Chapter VII

146. **Symmetry.**

An expression containing two or more letters is said to be *symmetrical* with respect to any two of them, when they can be interchanged without altering the value of the expression.

Thus, $a+b+c$ is symmetrical with respect to a and b for, on interchanging these letters, the expression becomes $b+a+c$.

An expression containing three or more letters is said to be symmetrical with respect to them when it is symmetrical with respect to any two of them.

Thus, $ab+bc+ca$ is symmetrical with respect to the letters a, b, and c; for if a and b be interchanged, the expression becomes $ba+ac+cb$, which is equal to $ab+bc+ca$.

And, in like manner, $ab+bc+ca$ is symmetrical with respect to b and c, and with respect to c and a.

147. **Cyclo-symmetry.**

An expression containing n letters, $a, b, c, \ldots, m, n$, is said to be *cyclo-symmetrical* with respect to them when, if a is substituted for b, b for c, $\ldots$, m for n, and n for a, the value of the expression is not changed.

The above is called a *cyclical* interchange of letters.

Thus, $(a-b)(b-c)(c-a)$ is cyclo-symmetrical with respect to a, b, and c; for if a is substituted for b, b for c, and c for a, it becomes $(c-a)(a-b)(b-c)$, which by the Commutative Law for Multiplication is equal to $(a-b)(b-c)(c-a)$.

148. Every expression which is symmetrical with respect to a set of letters is also cyclo-symmetrical with respect to them.

For since any two letters can be interchanged without altering the value of the expression, the condition for cyclo-symmetry will be satisfied.

But it is not necessarily true that an expression which is cyclo-symmetrical with respect to a set of letters is also symmetrical with respect to them; for it does not follow that any two letters can be interchanged without altering the value of the expression.

149. It follows from §§146 and 147 that, if two expressions are symmetrical or cyclo-symmetrical, the results obtained by adding, subtracting, multiplying, or dividing them are, respectively, symmetrical or cyclo-symmetrical.

150. **Applications.**

The principle of symmetry is often useful in abridging algebraic operations.

1. Expand $(a+b+c)^3$.
   We have, $(a+b+c)^3 = (a+b+c)(a+b+c)(a+b+c)$.

This expression is symmetrical with respect to a, b, and c (S 146), and of the third degree.

There are three possible types of terms of the third degree in a, b, and c; terms like a^3, terms like a^2b, and terms like abc.

It is evident that a^3 has the coefficient 1; and so, by symmetry, b^3 and c^3 have the coefficient 1.

It is evident that a^2b has the coefficient 3; and so, by symmetry, have b^2a, b^2c, c^2b, c^2a, and a^2c.

Let m denote the coefficient of abc.

Then,

$$(a+b+c)^3 = a^3+b^3+c^3+3\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right)+mabc.$$

To determine m, we observe that the above equation holds for all values of a, b, and c.

We may therefore let $a=b=c=1$

Then, $27 = 3+18+m$; and $m=6$.

Whence,

$$(a+b+c)^3 = a^3+b^3+c^3+3\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right)+6abc.$$

The above result may be written in a more compact form by representing the sum of terms of the same type by the symbol Σ; read *sigma*.[33]

Thus, $(\Sigma a)^3 = \Sigma a^3 + 3\Sigma a^2b + 6abc$

2. Expand $(x-y-z)^2+(y-z-x)^2+(z-x-y)^2$.

This expression is symmetrical with respect to x, y, and z and of the second degree.

The possible types of terms of the second degree in x, y, and z are terms like x^2, and terms like xy.

It is evident, by the law of §134, that x^2 has the coefficient 3; and so, by symmetry, have y^2 and z^2.

Let m denote the coefficient of xy. Then,

$$(x-y-z)^2+(y-z-x)^2+(z-x-y)^2 = 3\left(x^2+y^2+z^2\right)+m(xy+yz+zx).$$

To determine m, put $x=y=z=1$.

Then, $3 = 9+3m$, or $m=-2$.

Whence,

$$(x-y-z)^2+(y-z-x)^2+(z-x-y)^2 = 3\left(x^2+y^2+z^2\right)-2(xy+yz+zx)$$

[33] Sigma is the eighteenth letter of the Greek alphabet. In the system of Greek numerals, it has a value of 200. In general mathematics, uppercase Σ is used as an operator for summation. The lower case sigma character is σ.

3. Expand

$$(a+b+c)^3+(a+b-c)^3+(b+c-a)^3+(c+a-b)^3$$

The expression is symmetrical with respect to a, b, and c, and of the third degree.
The possible types of terms are terms like a^3, terms like a^2b, and terms like abc.
It is evident, by the law of §136, that a^3 has the coefficient 2; and so, by symmetry, have b^3 and c^3.
Also, by §136, a^2b has the coefficient $3+3+3-3$, or 6; and so, by symmetry, have b^2a, b^2c, c^2b, c^2a, and a^2c.
Again, abc has the coefficient $6-6-6-6$, or -12
Whence, $(a+b+c)^3+(a+b-c)^3+(b+c-a)^3+(c+a-b)^3$

$$=2\left(a^3+b^3+c^3\right)+6\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right)-12abc.$$

8.2 Examples

> "Mathematics is a hard thing to love. It has the unfortunate habit, like a rude dog, of turning its most unfavorable side towards you when you first make contact with it."
>
> —David Whiteland

Seeing an example before starting the exercises may be a needed gateway to success. However, you may also contact Ron Bannon ([redacted]) and request one-on-one assistance.

1. Expand $(a+b+c)^3$

Solution: This expression is symmetrical with respect to a, b, and c and the result will be degree three.

$$(a+b+c)^3 = (a+b+c)(a+b+c)(a+b+c)$$

There are three possible types of terms of the third degree in a, b, and c. We'll get something like this:

$$(a+b+c)^3 = a^3+b^3+c^3+3\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right)+mabc.$$

To determine m, just set $a=b=c=1$ (it's the easiest choice), and solve for m.

$$\begin{aligned}(a+b+c)^3 &= a^3+b^3+c^3+3\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right)+mabc\\ 27 &= 3+18+m\\ m &= 6\end{aligned}$$

Hence,

$$(a+b+c)^3 = a^3+b^3+c^3+3\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right)+6abc.$$

2. Expand $(x-y-z)^2+(y-z-x)^2+(z-x-y)^2$.

Solution: This expression is symmetrical with respect to x, y, and z, and of the second degree. The expanded form will be:

$$= 3\left(x^2+y^2+z^2\right)+m\left(xy+xz+yz\right)$$

To find m, I suggest you let $x = y = z = 1$.

$$\begin{aligned} 3 &= 9 + 3m \\ m &= -2 \end{aligned}$$

Hence,

$$(x-y-z)^2 + (y-z-x)^2 + (z-x-y)^2 = 3\left(x^2+y^2+z^2\right) - 2\left(xy+xz+yz\right)$$

3. Expand $(a+b+c)^3 + (a+b-c)^3 + (b+c-a)^3 + (c+a-b)^3$.

Solution: The expression is symmetrical with respect to a, b, and c, and of the third degree. The expanded form:

$$= 2\left(a^3+b^3+c^3\right) + 6\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right) + mabc$$

To solve for m you'll need to set $a = b = c = 1$, and you'll see that $m = -12$. Our expansion is:

$$= 2\left(a^3+b^3+c^3\right) + 6\left(a^2b+b^2a+b^2c+c^2b+c^2a+a^2c\right) - 12abc$$

8.3 Exercises

Although some problems are very challenging, you should be able to do most of the following exercises. Solutions and semi-detailed work appear after each problem as a way to validate your progress. Updates to this document will undoubtedly happen, and I encourage feedback[34] related to improving this document to help me improve the content. Work provided may not be complete, but it should be sufficient to understand the provided solutions.

Although I appreciate what Wells does in his text, I may be taking a different approach to solving *some* of the problems. Using your head and having options is needed if you plan to move forward in mathematics.

If, for whatever reason, you need me to show additional work, please do contact me, and I will make every effort to expand the provided work. You're not alone if you can't follow a particular solution. However, it would be best to have a good grasp of algebra to solve many of these problems. Wells' exercises follow directly from his lessons—that is, if you understand the examples given before starting the exercises, you should be able to work the entire problem set. Mistakes happen, so please do ask if you think something is amiss.

One final note, although my solutions may not be the *most elegant*, they are nonetheless the way I *initially* did the problem. Bear in mind that I *welcome more elegant* solutions and will certainly include them in future versions of this document, with *attributions*, of course!

1. In the expansion of an expression which is symmetrical with respect to a, b, and c, what are the possible types of terms of the fourth degree?

   Solution: a^4, b^4, c^4, a^3b, a^3c, ab^3, b^3c, a^2c^2, b^2c^2, a^2b^2, a^2bc, ab^2c, ac^3, bc^3, abc^2

2. If one term of an expression which is symmetrical with respect to a, b, and c, is $(2a-b-c)(2b-c-a)$, what are the others?

   Solution: $(2b-c-a)(2c-a-b)$ and $(2c-a-b)(2a-b-c)$.

3. Is the expression $a(b-c)^2+b(c-a)^2+c(a-b)^2$ symmetrical with respect to a, b, and c?

   Solution: Yes.

[34] Contact Ron Bannon at [redacted] if you have questions or concerns related to these problems. Try your best to indicate the source of your concern by *citing* the section and problem number.

4. Is the expression $\left(x^2 - y^2\right)^3 + \left(y^2 - z^2\right)^3 + \left(z^2 - x^2\right)^3$ symmetrical with respect to x, y, and z?

Solution: Yes.

5. Expand by the symmetrical method.

$$(a + b + c)^2$$

Solution: $a^2 + b^2 + c^2 + 2(ab + ac + bc)$

6. Expand by the symmetrical method.

$$(a + b + c + d)^2$$

Solution: $a^2 + b^2 + c^2 + d^2 + 2(ab + ac + bc + ad + bd + cd)$

7. Expand by the symmetrical method.

$$(x + y - z)^2 + (y + z - x)^2 + (z + x - y)^2$$

Solution: $3x^2 + 3y^2 + 3z^2 - 2(xy + xz + yz)$

8. Expand by the symmetrical method.

$$(2a - 3b - 4c)^2 + (2b - 3c - 4a)^2 + (2c - 3a - 4b)^2$$

Solution: $29a^2 + 29b^2 + 29c^2 - 4(ab + ac + bc)$

9. Expand by the symmetrical method.

$$(a + b + c - d)^2 + (b + c + d - a)^2 + (c + d + a - b)^2 + (d + a + b - c)^2$$

Solution: $4a^2 + 4b^2 + 4c^2 + 4d^2$

10. Expand by the symmetrical method.

$$(a + b + c + d)^3$$

Solution:

$$a^3 + b^3 + c^3 + d^3 + \\ 3\left(a^2b + ab^2 + a^2c + b^2c + ac^2 + bc^2 + a^2d + b^2d + c^2d + ad^2 + bd^2 + cd^2\right) \\ + 6\left(abc + abd + acd + bcd\right)$$

11. Expand by the symmetrical method.

$$(a + b + c)^3 + (a - b - c)^3 + (b - c - a)^3 + (c - a - b)^3$$

Solution: $24abc$

12. Expand by the symmetrical method.

$$(x + y - z)(y + z - x)(z + x - y)$$

Solution: $-\left(x^3 + y^3 + z^3\right) + \left(x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2\right) - 2xyz$

13. Expand by the symmetrical method.

$$\left[x^2 + y^2 + z^2 + 2(xy + yz + zx)\right]^2$$

Solution:

$$x^4 + y^4 + z^4 + 4\left(x^3y + xy^3 + x^3z + y^3z + xz^3 + yz^3\right) +$$
$$6\left(x^2y^2 + x^2z^2 + y^2z^2\right) + 12\left(x^2yz + xy^2z + xyz^2\right)$$

14. Expand by the symmetrical method.

$$(a+b+c)(a+b-c)(b+c-a)(a+c-b)$$

Solution: $-a^4 - b^4 - c^4 + 2\left(a^2b^2 + a^2c^2 + b^2c^2\right)$