# *Group Example: A Complex Circle*

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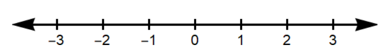
One interesting example of a group is the set of all complex numbers of magnitude 1. This is the circle of radius 1 in the complex plane centered at 0. This example shows it's possible to have an infinite group where all the elements have finite size.

## *The Complex Numbers*

The set of complex numbers $\mathbb{C}$ forms a group under addition. Given any two complex numbers $a + bi$ and $x + yi$ you can add them to get $(a + x) + (b + y)i$, which is also a complex number. For practice, check that the other group requirements are met: an identity element, associativity, and additive inverses.

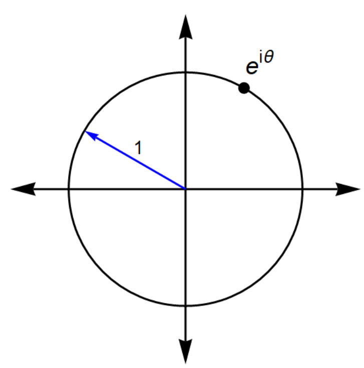
While the complex numbers $\mathbb{C}$ form a group under addition, they do *not* form a group under multiplication. The problem lies with the number 0. Zero does not have a multiplicative inverse because you cannot divide by zero. However, if you grab zero and toss it out the window, you DO get a group. The non-zero complex numbers are written as $\mathbb{C}^\times$ and this is a group under multiplication. (Be sure to convince yourself this is true!)

The complex numbers are a big collection of numbers and they contain a *lot* of smaller groups inside of it. For example, the group of integers $\mathbb{Z}$ under addition are contained inside $\mathbb{C}$. The real number line $\mathbb{R}$ is contained inside the complex plane, and it's also a group under addition.




## *The Unit Circle*

But today, we're going to talk about the unit circle in the complex plane. The phrase "unit circle" just means the circle with radius 1 centered at the origin. These complex numbers can be written in the form $\cos\theta + i \cdot \sin\theta$, and this can be written more compactly as $e^{i\theta}$.




Suppose you have two complex numbers on this circle. We'll call them $e^{i\alpha}$ and $e^{i\beta}$. Then if you multiply them together, you get $e^{i(\alpha+\beta)}$ which is also on the circle.

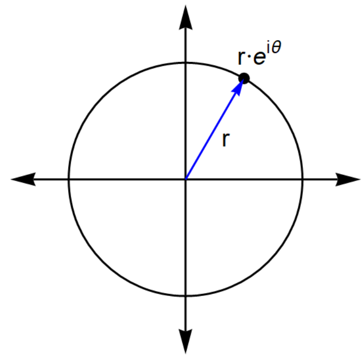
To be a group you need a few more things: an identity element, associativity, and inverses. The complex number $z = 1$ is our identity element, so we can check off that box. The inverse of $e^{i\alpha}$ is $e^{-i\alpha}$ because their product is

$$\begin{aligned} e^{i\alpha} \cdot e^{-i\alpha} &= e^{i\alpha - i\alpha} \\ &= e^0 \\ &= 1 \end{aligned}$$

And for practice, show that multiplication is also associative. That is,

$$e^{i\alpha}(e^{i\beta} \cdot e^{i\gamma}) = (e^{i\alpha} \cdot e^{i\beta}) \cdot e^{i\gamma}$$

## *Deep Dive - Are There Other Circles?*




You may be asking yourself, "Self, I wonder if any other circles are groups." Sadly, the answer is *no* and here's why: to be a group, you need a multiplicative identity. For the complex numbers under *multiplication*, the identity is 1. And there is only one circle centered at 0 that passes through the number 1, and that's the unit circle.

You may now be asking yourself, "Self, remain calm and skeptical. Does the circle *have* to be centered at 0? Is it possible there's a circle not centered at 0 that's a group under multiplication?

This is a fun problem to think about, and I'll give you a hint. For starters, the circle has to go through $z = 1$ because it needs to have a multiplicative identity. Also, it cannot go through $z = 0$ because 0 does not have a multiplicative inverse - you cannot divide by zero. So there is some point on the circle that's closest to $z = 0$, and another point that's farthest from $z = 0$.

So suppose you have a circle of radius $r$ on the complex plane that's *not* centered at $z = 0$. Now pick some point on this circle that's not a distance of 1 from the origin and let's call it $s \cdot e^{i\theta}$. Look at the powers of this number: $(s \cdot e^{i\theta})^n = s^n \cdot e^{in\theta}$. The $n^{th}$ power is a distance of $s^n$ from 0. What happens to this distance as $n$ gets bigger? It depends on the starting point. If the original point was *inside* the circle, then the powers will get closer and closer to 0. And if the original point was *outside* of the circle, the powers will get further and further away from zero. But earlier we noted there was a closest point and farthest point to 0. These powers will eventually get closer to $z = 0$ than any number, or farther from $z = 0$ than any number. It just doesn't work!

## *Advanced Conclusion*

The circle in the complex plane of radius 1 and centered at $z = 0$ is called "the unit circle." The mathematical notation for this is $S^1$. This is because it's a 1-dimensional "sphere". You will study $S^2, S^3, \ldots S^n$ for higher dimensional spheres which are called "hyperspheres." $S^1$ is important because it's an example of a group, a topological space, a metric space, and even a manifold. It's an important concept to have in your back pocket because the more you study mathematics, the you will see this as an example of more advanced concepts.