Isomorphism Example

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In this example, we will show that the group of real numbers under addition $\langle \mathbb{R}, + \rangle$ is isomorphic to the group of positive real numbers under multiplication $\langle \mathbb{R}^{+}, \cdot \rangle$.

Quick Reminder

Suppose you have two groups, $\langle G, * \rangle$ and $\langle K, \diamond \rangle$. A homomorphism f is a function $f : G \to K$ that satisfies one essential property:

$$f(x * y) = f(x) \diamond f(y)$$

Homomorphism:
$f(a \cdot b) = f(a) \cdot f(b)$

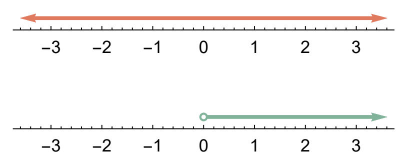
An *isomorphism* is a homomorphism that is also "one-to-one and onto." Another way to say this is that an isomorphism is a homomorphism that's also a *bijection*.

isomorphism =
homomorphism + bijection

The Two Groups

The first group G is the set of all real numbers under addition. We write this group as $G = \langle \mathbb{R}, + \rangle$ to highlight both the elements (real numbers) and the operation (addition).

The second group K is the set of all *positive* real numbers under multiplication. Like before, we write this group as $K = \langle \mathbb{R}^{+}, \cdot \rangle$ to make clear the elements and group operation.



The Isomorphism

If I were to just tell you the isomorphism, it would seem like I was pulling the answer out of my hat. Instead, I'd like to share with you a little bit of mathematical history. This example has a connection to an important development in mathematics – *logarithms*. When logs were developed, they simplified the process of multiplying two numbers. And in a time without calculators, this was a *big* time saver.

Before calculators, if you had to multiply two large numbers together, you would look up their logarithms in a table, add the two logs, then find the corresponding number to the sum in the log table. This number would be the product! Logarithms converted the problem of multiplying (slow) into adding (fast).

So multiplying positive real numbers and adding real numbers have long been connected by logarithms and their inverse functions *exponentials*.

John Napier
Credited with developing logarithms

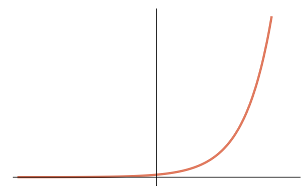
An Isomorphism

The function $f : G \to K$ defined by $f(x) = e^x$ is *AN* isomorphism between these two groups. (There is often more than one isomorphism between groups.) Let's first check that it's a homomorphism:

$$\begin{aligned} f(x+y) &= e^{x+y} \\ &= e^x \cdot e^y \\ &= f(x) \cdot f(y) \end{aligned}$$

So f is a homomorphism. But is it an *isomorphism*?

If you look at the graph of $f(x) = e^x$, you will see that it is one-to-one (it passes the horizontal line test) and it "onto" because the range of $f(x)$ is $(0, \infty)$. Because $f(x)$ is both a homomorphism and a bijection, it's an *isomorphism*.



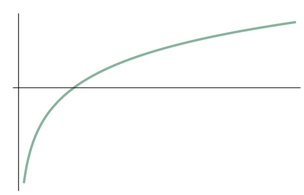
$y = e^x$

Let's dig a little deeper

IF WE SWITCH MENTAL GEARS TO CALCULUS MODE, this function is also a continuous function. In fact, you can repeatedly take the derivative of e^x, so we call it a *smooth* function. These two groups have an algebraic connection (an isomorphism) and an analytic connection (a smooth function). If you were to continue down this rabbit hole, you would end up talking about things called *Lie Groups* or *Abelian Varieties*. But we'll save that story for another day.

The Inverse Isomorphism

The inverse of the isomorphism $f(x) = e^x$ is $h : \mathbb{R}^+ \to \mathbb{R}$ defined by $h(x) = \log x$. Like before, let's first check that this is a homomorphism:



$y = \log x$

$$\begin{aligned} h(x \cdot y) &= \log(x \cdot y) \\ &= \log x + \log y \\ &= h(x) + h(y) \end{aligned}$$

So the natural logarithm is a *homomorphism* from the group $K = \langle \mathbb{R}^+, \cdot \rangle$ to the group of real numbers under addition $G = \langle \mathbb{R}, + \rangle$. Like we saw with $f(x)$, the log function $g(x)$ is one-to-one (an injection) with a range of all real numbers. So $g(x)$ is an isomorphism.