Algebra
Seeing Structures in Expressions
Annuities
High School

Rationale

- At some point in their lives, students will likely have to make a series of fixed payments over a period of time, such as mortgage payments or car payments. These types of loans are called annuities, and the present value of an annuity can be modeled with an exponential function, which relates the present value of the loan to the amount of the borrower's monthly payment, the interest rate of the loan, and the total number of pay periods specified in the loan. In this lesson, students will "buy" a car and will use this formula to calculate the amount of their monthly car payments. As students will discover, mastering this skill may give them an advantage when they actually do buy their first car (or other expensive purchase) in the real world.

Goal

- To have students increase their familiarity with rewriting and using exponential functions to solve real-world problems

Standards

- **A-SSE.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★
  c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{\frac{1}{12}})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*
- **A-SSE.4** Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★ (See **Extension Activities**.)

Objectives

- Students will use the formula for a present value annuity to calculate the amount of a monthly car payment, given the present value of the car, an interest rate, and a term length of an automobile loan.
- Students will use the order of operations and the properties of exponents to rewrite an exponential expression associated with an exponential function.
- Students will derive the formula for the sum of a finite geometric series.(See Extension Activities.)
- Students will use the formula for the sum of a finite geometric series to solve a real-world problem.(See Extension Activities.)

Materials

- Graphing calculators and/or spreadsheet software
- Computers with internet access (optional)

Procedures

- Begin the lesson by asking students what types of loans they are familiar with. Students will probably mention automobile loans or loans for purchasing a home, also known as a mortgage. Ask students, "What is a loan?" Give them time to think and form a reasonable explanation. Then call on several students to share their thoughts and ideas. Students may mention some of the following ideas:
  - A person may get a loan to help pay for big-ticket purchases, such as automobiles, homes, boats, jewelry, etc.
  - A person may get a loan when he or she doesn't have enough money to completely cover the cost of a purchase. (Or, similarly, a person may get a loan when he or she doesn't want to spend a large sum of their money at one time.)
  - A person can get a loan from a financial institution, such as a bank or credit union, or from an individual, such as a family member or friend.
  - A person usually pays back a loan in equal installments over a period of time.

  Allow students' thoughts and ideas to guide the conversation, but make sure they walk away understanding the following about a loan:
  - A loan is a legally binding agreement in which a lender gives a borrower a certain amount of money. The borrower is expected to pay the lender back in full, usually plus interest, at a later date.
- Say, "Today, you are going to take out a loan to buy your dream car." For the sake of simplicity, tell all students to assume that they are going to purchase their cars with no money down, meaning that they will finance the entire purchase price of their cars. Then instruct students to estimate the cost of a car they want to buy. Assign each student an interest rate and a term length for their automobile loan. Some common interest rates include 1%, 1.9%, 2.9%, 5.25%, 5.5%, 6%, 6.5%, 7.25%, and 7.5%; common term lengths for automobile loans are 3 years, 4 years, 5 years, and 6 years.
- Explain that students will work to find the amount of their monthly car payment, the total amount they will have paid for the car once the loan is paid off, and the total amount of interest they will have paid over the term of the loan. However, before they figure out the precise values, ask students to *guesstimate* all three amounts. Give students time to think and make some basic calculations, if they wish, in order to better estimate each amount. Then have students record their guesstimates on a separate sheet of paper, which they will come back to later in the lesson.
- Students have likely never made these types of calculations before, so before you have them solve a problem using their own values, you will probably

want to walk them through at least the first few steps of a similar problem, such as the following:

Suppose you purchase a car for $23,000 and make equal payments each month for 5 years. You are charged an annual interest rate of 3% compounded monthly in your loan.

- Find the amount of each monthly payment.
- Find the total amount you will have paid for the car once you pay off the loan.
- Find the total amount of interest you will have paid over the term of the loan.

- Begin walking students through the process. Start by giving them the general formula $P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$, which they will use to find the amount of their monthly payments. Point out to students that this formula is an exponential function containing a negative exponent. Also take some time to explain what each variable represents: *P* represents the present value of the car (the cost of the car), *R* represents the periodic payment (the amount of the monthly payment), *i* represents the interest rate *per period* (expressed as a decimal), and *n* represents the total number of periods. Your work may resemble the following:

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$23{,}000 = R\left(\frac{1-\left(1+\frac{0.03}{12}\right)^{-5\cdot 12}}{\frac{0.03}{12}}\right)$	Substitute.
$23{,}000 = R\left(\frac{1-(1+0.0025)^{-60}}{0.0025}\right)$	Simplify.
$23{,}000 = R\left(\frac{1-(1.0025)^{-60}}{0.0025}\right)$	Simplify.
$23{,}000 = R\left(\frac{1-\frac{1}{1.0025^{60}}}{0.0025}\right)$	Use the definition of negative exponents.
$23{,}000 \approx R\left(\frac{1-\frac{1}{1.161616782}}{0.0025}\right)$	Simplify.
$23{,}000 \approx R\left(\frac{1-0.860869106}{0.0025}\right)$	Simplify.
$23{,}000 \approx R\left(\frac{0.139130894}{0.0025}\right)$	Simplify.
$23{,}000 \approx R(55.65235769)$	Divide each side by 55.65235769.
$R \approx 413.2798853$	

So, the monthly payment will be about $413.28.

- Once you find the amount of the monthly payment, students should be able to find the total amount you will have paid for the car once you pay off the loan.

Multiply the amount of the monthly car payment, 413.28, by the total number of payments, 60.

413.28(60) = 24,796.80

- Students should also be able to find the total amount of interest you will have paid over the term of the loan. Subtract the cost of the car from the total amount you will have paid for the car.

  24,796.80 – 23,000 = 1796.80

- Give students time to perform the calculations using their own values. Then have students compare their actual values to their guesstimates from earlier. Take some time to discuss how students' values would vary with different interest rates and/or different term lengths.
- Closing: Have students work independently to solve the following problem: Suppose you purchase a home for $125,000 and make equal payments each month for 30 years. You are charged an annual interest rate of 5% compounded monthly in your loan.
  - Find the amount of each monthly payment.
  - Find the total amount you will have paid for the home once you pay off the loan.
  - Find the total amount of interest you will have paid over the term of the loan.

  Once students have solved the problem, have them pair up with a partner and compare answers.

Teacher tips

- Before beginning this lesson, students should be familiar with the following skills and concepts:
  - Writing percents as decimals
  - Properties of exponents
  - Exponential functions and models
- Students may have difficulty understanding the meaning of the phrase *compounded monthly* in example. If so, explain that when interest is compounded monthly, the amount of interest is calculated each month based on the how much of the loan the borrower has yet to pay.
- You may want to point out to students that the formula for finding the amount of the monthly car payments is called the formula for the present value of an annuity. *Annuity* refers to a finite series of fixed payments over a specific period of time.
- You may also want to point out to students that the formula for finding the amount of the monthly car payments is an example of a literal formula. So, if students prefer, they can first solve the formula for R and then substitute and solve for R.
- To engage students in the activity, you may wish to have them randomly select their interest rates and term lengths from numbers in a hat. You will need to prepare the rates and terms lengths in advance of the activity.
- Students may need to be reminded to convert from years to months in their formulas. A 5-year term, for example, must be changed to $5 \times 12 = 60$ for the formula.

Extension activities

- If time permits, you may wish to let students research the actual costs of their dream cars. Use the Internet to give students a chance to find the precise amounts of their loans. Ensure that students consider additional costs for the car, such as closing fees, add-on features, or an extended warranty.
- Have students repeat a similar activity from the lesson by using the maximum monthly payment instead of the cost of the large-scale purchase. First, have them determine how much they can actually afford to pay per month for a car. (Students can consider what annual income might allow them to afford a particular monthly payment for a car.) Then have them use the formula to determine the maximum purchase price of a car.
- Have students research and create an amortization table that shows how much of each monthly car payment goes toward interest and how much goes toward principal. You may also want students to research how to use spreadsheet software and/or a graphing calculator to create an amortization table.
- The formula for the present value of an annuity actually represents the sum of a finite geometric series with *n* terms, a common ratio of $\frac{1}{1+i}$, and a first term of $\frac{R}{1+i}$. Have students work together in small groups and use this information to determine how much a state lottery would need to invest today in order to pay someone a $1 million annuity.